ITERATIVE SCHEMES WITH SOME CONTROL CONDITIONS FOR A FAMILY OF FINITE NONEXPANSIVE MAPPINGS IN BANACH SPACES

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The iterative schemes with some control conditions for a family of finite nonexpansive mappings are established in a Banach space. The main theorem improves results of Jung and Kim (also Bauschke). Our results also improve the corresponding results of Cho et al., Shioji and Takahashi, Xu, and Zhou et al. in certain Banach spaces and of Lions, O'Hara et al., and Wittmann in a framework of a Hilbert space, respectively.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E and let $T_1,\dots,T_N$ be nonexpansive mappings from C into itself (recall that a mapping $T: C \to C$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$).

We consider the iterative scheme. For a positive integer N, nonexpansive mappings $T_1, T_2,\dots,T_N$, $a, x_0 \in C$, and $\lambda_n \in (0,1]$,

$$x_{n+1} = \lambda_{n+1}a + (1-\lambda_{n+1})T_{n+1}x_n, \quad n \ge 0. \tag{1.1}$$

In 1967, Halpern [7] firstly introduced the iteration scheme (1.1) for $a = 0$, $N = 1$ (i.e., he considered only one mapping T); see also Browder [2]. He showed that the conditions

$$\lim_{n\to\infty} \lambda_n = 0, \tag{1.2}$$

$$\sum_{n=1}^{\infty} \lambda_n = \infty \qquad \text{or, equivalently,} \qquad \prod_{n=1}^{\infty}(1-\lambda_n) := \lim_{n\to\infty}\prod_{k=1}^{n}(1-\lambda_k) = 0 \tag{1.3}$$

are necessary for convergence of the iterative scheme (1.1) to a fixed point of T. Ten years later, Lions [11] investigated the general case in Hilbert spaces under (1.2), (1.3), and

$$\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0 \tag{1.4}$$

on the parameters. However, Lions' conditions on the parameters were more restrictive and did not include the natural candidate $\lambda_n = 1/(n+1)$. In 1980, Reich [16] gave the

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iterative scheme (1.1) for $N = 1$ in the case when E is uniformly smooth and $\lambda_n = 1/n^s$ with $0 < s < 1$.

In 1992, Wittmann [20] studied the iteration scheme (1.1) for $N = 1$ in the case when E is a Hilbert space and $\{\lambda_n\}$ satisfies (1.2), (1.3), and

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty. \tag{1.5}$$

In 1994, Reich [17] obtained a strong convergence of the iterative scheme (1.1) for $N = 1$ with two necessary and decreasing conditions on parameters for convergence in the case when E is uniformly smooth with a weakly continuous duality mapping. In 1996, Bauschke [1] improved results of Wittmann [20] to finitely many mappings with additional condition on the parameters

$$\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty, \tag{1.6}$$

where $T_n := T_{n \bmod N}$, $N > 1$. He also provided an algorithmic proof which has been used successfully, with modifications, by many authors [4, 13, 18, 22]. In 1997, Jung and Kim [9] extended Bauschke's result to a Banach space and Shioji and Takahashi [19] improved Wittmann's result to a certain Banach space. Shimizu and Takahashi [18], in 1997, dealt with the above iterative scheme with the necessary conditions on the parameters and some additional conditions imposed on the mappings in a Hilbert space. Recently, O'Hara et al. [13] generalized the result of Shimizu and Takahashi [18] and also complemented a result of Bauschke [1] by imposing a new condition on the parameters

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1 \qquad \text{or, equivalently,} \qquad \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+N}}{\lambda_{n+N}} = 0 \tag{1.7}$$

in the framework of a Hilbert space. Xu [22] also studied some control conditions of Halpern's iterative sequence for finite nonexpansive mappings in Hilbert spaces. Very recently, Jung [8] extended the results of O'Hara et al. [13] to a Banach space. By using the Banach limit as in [19, 23], Zhou et al. [24] also provided the strong convergence of the iterative scheme (1.1) in certain Banach spaces with the weak asymptotically regularity.

In this paper, we consider the perturbed control condition

$$|\alpha_{n+N} - \alpha_n| \le \circ(\alpha_{n+N}) + \sigma_n, \tag{1.8}$$

where $\sum_{n=1}^{\infty} \sigma_n < \infty$, and prove a strong convergence of the iterative scheme (1.1) in a uniformly smooth Banach space with a weakly sequentially continuous duality mapping. Our results improve the corresponding results of Bauschke [1], Jung [8], Jung and Kim [9], O'Hara et al. [13], Zhou et al. [24] along with Cho et al. [3], Lions [11], Shioji and Takahashi [19], Wittmann [20], Xu [21, 23], and others.

2. Preliminaries and lemmas

First, we mention the relations between conditions (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), and give an example satisfying the perturbed control condition (1.8).

In general, the control conditions (1.6) and (1.7) are not comparable (coupled with conditions (1.2) and (1.3)), that is, neither of them implies the others as in the following examples.

Example 2.1. Consider the control sequence $\{\alpha_n\}$ defined by

$$\alpha_n = \begin{cases} \dfrac{1}{n^s} & \text{if } n \text{ is odd,} \\ \dfrac{1}{n^s} + \dfrac{1}{n^t} & \text{if } n \text{ is even,} \end{cases} \tag{2.1}$$

with $1/2 < s < t \leq 1$. Then $\{\alpha_n\}$ satisfies conditions (1.2), (1.3), and (1.7), but it fails to satisfy condition (1.6), where N is odd.

Example 2.2. Take two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that
(i) $m_1 = 1$, $m_k < n_k$, and $\max\{4n_k, n_k + N\} < m_{k+1}$ for $k \geq 1$,
(ii) $\sum_{i=m_k}^{n_k}(1/\sqrt{i}) > 1$ for $k \geq 1$.
Define a sequence $\{\mu_n\}$ by

$$\mu_i = \begin{cases} \dfrac{1}{\sqrt{i}} & \text{if } m_k \leq i \leq n_k,\ k \geq 1, \\ \dfrac{1}{2\sqrt{n_k}} & \text{if } n_k < i < m_{k+1},\ k \geq 1. \end{cases} \tag{2.2}$$

Then $\{\mu_n\}$ is decreasing and $\mu_n \to 0$ as $n \to \infty$. Hence conditions (1.2) and (1.6) are satisfied. Noting that

$$\sum_{n=1}^{\infty} \mu_n \geq \sum_{k=1}^{\infty} \sum_{i=m_k}^{n_k} \mu_i = \infty, \tag{2.3}$$

then we see that condition (1.3) is also satisfied. On the other hand, we have

$$\frac{\mu_{n_k}}{\mu_{n_k+N}} = 2, \quad k \geq 1, \tag{2.4}$$

which shows that condition (1.7) is not satisfied.

Example 2.3 (Xu [22]). Consider the control sequence $\{\alpha_n\}$ defined by

$$\alpha_n = \begin{cases} \dfrac{1}{\sqrt{n}} & \text{if } n \text{ is odd,} \\ \dfrac{1}{\sqrt{n} - 1} & \text{if } n \text{ is even.} \end{cases} \tag{2.5}$$

Then $\{\alpha_n\}$ satisfies (1.7), but it fails to satisfy (1.6).

Example 2.4. Take $\{\alpha_n\}$ and $\{\mu_n\}$ as in the above Examples 2.1 and 2.2. Define a sequence $\{\lambda_n\}$ by

$$\lambda_n = \alpha_n + \mu_n \tag{2.6}$$

for all $n \geq 1$. Then $\{\lambda_n\}$ satisfies conditions (1.2), (1.3), and

$$\left|\lambda_{n+N} - \lambda_n\right| \leq \circ(\lambda_{n+N}) + \sigma_n, \tag{2.7}$$

where $\sum_{n=1}^{\infty} \sigma_n < \infty$, but it fails to satisfy both conditions (1.6) and (1.7). For the case $N = 1$, we also refer to [3].

Example 2.5. Let $\{\alpha_n\}$ satisfy (1.2), (1.3), not (1.6), (1.7) and let $\{\mu_n\}$ be (1.2), (1.3), (1.6), not (1.7). Assume that

$$\lim_{n\to\infty} \frac{\alpha_n}{\mu_n} = 0, \tag{2.8}$$

and define a sequence $\{\lambda_n\}$ by

$$\lambda_n = \alpha_n + \mu_n \tag{2.9}$$

for all $n \geq 1$. Then $\{\lambda_n\}$ satisfies conditions (1.2), (1.3), not (1.6), (1.7), and (1.8).

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.10}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (2.10) is attained uniformly for $(x, y) \in U \times U$.

The (normalized) *duality mapping* J from E into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\} \tag{2.11}$$

for each $x \in E$. It is single valued if and only if E is smooth. It is also well known that if E has a uniformly Fréchet differentiable norm, J is uniformly continuous on bounded subsets of E. Suppose that J is single valued. Then J is said to be *weakly sequentially continuous* if, for each $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J(x_n) \overset{*}{\rightharpoonup} J(x)$.

We need the following lemma for the proof of our main results, which was given by Jung and Morales [10]. It is actually Petryshyn's [15, Lemma 1].

Lemma 2.6. *Let X be a real Banach space and let J be the normalized duality mapping. Then, for any given $x, y \in X$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{2.12}$$

for all $j(x+y) \in J(x+y)$.

A Banach space E is said to satisfy *Opial's condition* [14] if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{2.13}$$

for all $y \in E$ with $y \neq x$. It is well known that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition.

Recall that a mapping T defined on a subset C of a Banach space E (and taking values in E) is said to be *demiclosed* if, for any sequence $\{u_n\}$ in C, the following implication holds:

$$u_n \rightharpoonup u, \qquad \lim_{n\to\infty} \|Tu_n - w\| = 0 \tag{2.14}$$

implies that

$$u \in C, \qquad Tu = w. \tag{2.15}$$

The following lemma can be found in [5, page 108].

Lemma 2.7. *Let E be a reflexive Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Suppose that $T : C \to E$ is a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping.*

Let C be a nonempty closed convex subset of E. A mapping Q of C into C is said to be a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is range of Q. Let D be a subset of C and let Q be a mapping of C into D. Then Q is said to be *sunny* if each point on the ray $\{Qx + t(x - Qx) : t > 0\}$ is mapped by Q back onto Qx; in other words,

$$Q\big(Qx + t(x - Qx)\big) = Qx \tag{2.16}$$

for all $t \geq 0$ and $x \in C$. A subset D of C is said to be a *sunny nonexpansive retraction* of C if there exists a sunny nonexpansive retraction of C onto D. For more details, we refer to [6].

The following lemma is well known (cf. [6, page 48]).

Lemma 2.8. *Let C be a nonempty closed convex subset of a smooth Banach space E, let D be a subset of C, let $J : E \to E^*$ be the duality mapping of E, and let $Q : C \to D$ be a retraction. Then the following are equivalent:*

(a) $\langle x - Qx, J(y - Qx)\rangle \leq 0$ *for all $x \in C$ and $y \in D$;*

(b) $\|Qz - Qw\|^2 \leq \langle z - w, J(Qz - Qw)\rangle$ *for all z and w in C;*

(c) *Q is both sunny and nonexpansive.*

Finally, we need the following lemma, which is essentially Liu's [12, Lemma 2]. For the sake of completeness, we give the proof.

Lemma 2.9. *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n\beta_n + \gamma_n, \quad n \ge 0, \tag{2.17}$$

where $\{\lambda_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *satisfy the following conditions:*

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty}\lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1-\lambda_n) := \lim_{n\to\infty}\prod_{k=0}^{n}(1-\lambda_k) = 0$;

(ii) $\limsup_{n\to\infty}\beta_n \le 0$; *or*

(iii) $\sum_{n=1}^{\infty}\lambda_n\beta_n < \infty$;

(iv) $\gamma_n \ge 0$ $(n \ge 0)$, $\sum_{n=0}^{\infty}\gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Proof. First assume that (i), (ii), and (iv) hold. For any $\varepsilon > 0$, let $N \ge 1$ be an integer such that

$$\beta_n < \varepsilon, \quad \sum_{n=N}^{\infty}\gamma_n < \varepsilon, \quad n > N. \tag{2.18}$$

By using (2.17) and straightforward induction, we obtain

$$s_{n+1} \le \left(\prod_{k=N}^{n}(1-\lambda_k)\right)s_N + \left(1 - \prod_{k=N}^{n}(1-\lambda_k)\right)\varepsilon + \sum_{k=N}^{n}\gamma_k \tag{2.19}$$

for any $n > N$. Then conditions (i), (ii), and (iv) imply that $\limsup_{n\to\infty} s_n \le 2\varepsilon$.

Next, assume that (i), (iii), and (iv) hold. Then, repeatedly using (2.17), we have

$$s_{n+1} \le \prod_{k=m}^{n}(1-\lambda_k)s_m + \sum_{k=m}^{n}\lambda_k\beta_k + \sum_{k=m}^{n}\gamma_n \tag{2.20}$$

for any $n > m$. Letting in (2.20) first $n \to \infty$ and then $m \to \infty$, we obtain $\limsup_{n\to\infty} s_n \le 0$. $\square$

3. Main results

Using the perturbed control condition, we study the strong convergence result for a family of finite nonexpansive mappings in a Banach space.

We consider N mappings $T_1, T_2, \ldots, T_N$. For $n > N$, set $T_n := T_{n \bmod N}$, where $n \bmod N$, is defined as follows: if $n = kN + l$, $0 \le l < N$, then

$$n \bmod N := \begin{cases} l & \text{if } l \neq 0, \\ N & \text{if } l = 0. \end{cases} \tag{3.1}$$

We will use $\mathrm{Fix}(T)$ to denote the fixed point set of T, that is,

$$\mathrm{Fix}(T) := \{x \in C : Tx = x\}. \tag{3.2}$$

THEOREM 3.1. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$ and let C be a nonempty closed convex subset of E. Let $T_1,\dots,T_N$ be nonexpansive mappings from C into itself with $F := \cap_{i=1}^N \operatorname{Fix}(T_i) \neq \emptyset$ and*

$$F = \operatorname{Fix}(T_N \cdots T_1) = \operatorname{Fix}(T_1 T_N \cdots T_3 T_2) = \cdots = \operatorname{Fix}(T_{N-1} T_{N-2} \cdots T_1 T_N). \tag{3.3}$$

Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies conditions (1.2), (1.3), and (1.8). Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto F.

Proof. As in the proof of [1, Theorem 1], we proceed with the following steps.
Step 1. $\|x_n - z\| \le \max\{\|x_0 - z\|, \|a - z\|\}$ for all $n \ge 0$ and all $z \in F$ and so $\{x_n\}$ is bounded.

We use an inductive argument. The result is clearly true for $n = 0$. Suppose the result is true for n. Let $z \in F$. Then, since T_{n+1} is nonexpansive,

$$\begin{aligned} \|x_{n+1} - z\| &= \|\lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n - z\| \\ &\le \lambda_{n+1} \|a - z\| + (1 - \lambda_{n+1}) \|T_{n+1} x_n - z\| \\ &\le \lambda_{n+1} \|a - z\| + (1 - \lambda_{n+1}) \|x_n - z\| \\ &\le \lambda_{n+1} \max\{\|x_0 - z\|, \|a - z\|\} + (1 - \lambda_{n+1}) \max\{\|x_0 - z\|, \|a - z\|\} \\ &= \max\{\|x_0 - z\|, \|a - z\|\}, \end{aligned} \tag{3.4}$$

and $\|x_n\| \le \|x_n - z\| + \|z\| \le \max\{\|x_0 - z\|, \|a - z\|\} + \|z\|$.
Step 2. $\{T_{n+1} x_n\}$ is bounded. For all $n \ge 0$ and $z \in F$, since

$$\|T_{n+1} x_n\| \le \|T_{n+1} x_n - z\| + \|z\| \le \|x_n - z\| + \|z\| \le \max\{\|x_0 - z\|, \|a - z\|\} + \|z\| \tag{3.5}$$

for all $n \ge 0$ and $z \in F$, it follows that $\{T_{n+1} x_n\}$ is bounded.
Step 3. $\lim_{n\to\infty} \|x_{n+1} - T_{n+1} x_n\| = 0$. Indeed, since

$$\|x_{n+1} - T_{n+1} x_n\| = \lambda_{n+1} \|a - T_{n+1} x_n\| \le \lambda_{n+1} (\|a\| + \|T_{n+1} x_n\|) \le \lambda_{n+1} M \tag{3.6}$$

for some M, we also have

$$\lim_{n\to\infty} \|x_{n+1} - T_{n+1} x_n\| = 0. \tag{3.7}$$

Step 4. $\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0$. By Step 2, there exists a constant $L > 0$ such that for all $n \ge 1$,

$$\|a - T_{n+1} x_n\| \le L. \tag{3.8}$$

Since for all $n \geq 1$, $T_{n+N} = T_n$, we have

$$\begin{aligned} \|x_{n+N} - x_n\| &= \|(\lambda_{n+N} - \lambda_n)(a - T_n x_{n-1}) + (1 - \lambda_{n+N})(T_n x_{n+N-1} - T_n x_{n-1})\| \\ &\leq L\|\lambda_{n+N} - \lambda_n\| + (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| \\ &= (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| + \|\lambda_{n+N} - \lambda_n\| L \\ &\leq (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| + (\circ(\lambda_{n+N}) + \sigma_n)L. \end{aligned} \tag{3.9}$$

By taking $s_{n+1} = \|x_{n+N} - x_n\|$, $\lambda_{n+N} = \alpha_n$, $\circ(\lambda_{n+N})L = \alpha_n\beta_n$, and $\gamma_n = \sigma_n L$, we have

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \tag{3.10}$$

and, by Lemma 2.9,

$$\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0. \tag{3.11}$$

Step 5. $\lim_{n\to\infty} \|x_n - T_{n+N},\ldots,T_{n+1}x_n\| = 0$. By the proof in [1] with Step 4, we can obtain this fact and so its proof is omitted.
Step 6. $\limsup_{n\to\infty}\langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle \leq 0$. Let a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ be such that

$$\lim_{j\to\infty} \langle a - Q_F a, J(x_{n_j+1} - Q_F a)\rangle = \limsup_{n\to\infty} \langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle. \tag{3.12}$$

We assume (after passing to another subsequence if necessary) that $n_j + 1 \bmod N = i$ for some $i \in \{1,\ldots,N\}$ and that $x_{n_j+1} \rightharpoonup x$. From Step 5, it follows that $\lim_{j\to\infty} \|x_{n_j+1} - T_{i+N}\cdots T_{i+1}x_{n_j+1}\| = 0$. Hence, by Lemma 2.7, we have $x \in \operatorname{Fix}(T_{i+N}\cdots T_{i+1}) = F$.

On the other hand, since E is uniformly smooth, F is a sunny nonexpansive retraction of C (cf. [6, page 49]). Thus, by weakly sequentially continuity of duality mapping J and Lemma 2.8, we have

$$\begin{aligned} &\limsup_{n\to\infty} \langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle \\ &\quad = \lim_{j\to\infty} \langle a - Q_F a, J(x_{n_j+1} - Q_F a)\rangle = \langle a - Q_F a, J(x - Q_F a)\rangle \leq 0. \end{aligned} \tag{3.13}$$

Step 7. $\lim_{n\to\infty} \|x_n - Q_F a\| = 0$. By using (1.1), we have

$$(1 - \lambda_{n+1})(T_{n+1}x_n - Q_F a) = (x_{n+1} - Q_F a) - \lambda_{n+1}(a - Q_F a). \tag{3.14}$$

Applying Lemma 2.6, we obtain

$$\begin{aligned} \|x_{n+1} - Q_F a\|^2 &\leq (1 - \lambda_{n+1})^2\|T_{n+1}x_n - Q_F a\|^2 + 2\lambda_{n+1}\langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle \\ &\leq (1 - \lambda_{n+1})\|x_n - Q_F a\|^2 + 2\lambda_{n+1}\beta_n, \end{aligned} \tag{3.15}$$

where $\beta_n = \langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle$. By Step 6, $\limsup_{n\to\infty}\beta_n \leq 0$. Now, if we define $\delta_n = \max\{0,\beta_n\}$, then $\delta_n \to 0$ as $n \to \infty$ and so (3.15) reduces to

$$\begin{aligned} \|x_{n+1} - Q_F a\|^2 &\leq (1 - \lambda_{n+1})^2\|T_{n+1}x_n - Q_F a\|^2 + 2\lambda_{n+1}\delta_n \\ &\leq (1 - \lambda_{n+1})\|x_n - Q_F a\|^2 + \circ(\lambda_{n+1}). \end{aligned} \tag{3.16}$$

Thus it follows from Lemma 2.9 with $\gamma_n = 0$ that Step 7 holds. This completes the proof. □

As an immediate consequence, we have the following corollary.

Corollary 3.2. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let $T_1,\dots,T_N$ be nonexpansive mappings from C into itself with $F := \cap_{i=1}^{N} \operatorname{Fix}(T_i) \neq \emptyset$ and*

$$F = \operatorname{Fix}(T_N \cdots T_1) = \operatorname{Fix}(T_1 T_N \cdots T_3 T_2) = \cdots = \operatorname{Fix}(T_{N-1} T_{N-2} \cdots T_1 T_N). \quad (3.17)$$

Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies conditions (1.2), (1.3), and (1.8). Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to $P_F a$, where P is the nearest point projection of C onto F.

Proof. Note that the nearest point projection P of C onto F is a sunny nonexpansive retraction. Thus the result follows from Theorem 3.1. □

Corollary 3.3. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$ and let C be a nonempty closed convex subset of E. Let T be nonexpansive mappings from C into itself with $F = \operatorname{Fix}(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies conditions (1.2), (1.3), and (1.8). Then the iterative sequence $\{x_n\}$ defined by (1.1) with $T = T_1$ $(N = 1)$ converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto $F = \operatorname{Fix}(T)$.*

Remark 3.4. Since condition (1.8) includes conditions (1.4), (1.5), (1.6), and (1.7) as special cases, our main results unify and improve the corresponding results obtained by Bauschke [1], Jung [8], Jung and Kim [9], O'Hara et al. [13] for $N > 1$ and by Cho et al. [3], Lions [11], Shioji and Takahashi [19], Wittmann [20], Xu [21, 23], and others for $N = 1$, respectively.

Remark 3.5. (1) Our proof lines of Theorem 3.1 are different from those of Zhou et al. [24], in which, as in [19, 23], they utilized the concept of Banach's limit along with the weak asymptotically regularity and Reich's result [16] to prove their main results.

(2) Corollary 3.3 does not also use Reich's result [16] in comparison with those of Cho et al. [3], Shioji and Takahashi [19], and Xu [21].

Let D be a subset of a Banach space E. Recall that a mapping $T : D \to E$ is said to be firmly nonexpansive if for each x and y in D, the convex function $\phi : [0,1] \to [0,\infty)$ defined by

$$\phi(s) = \|(1-s)x + sTx - ((1-s)y + sTy)\| \quad (3.18)$$

is nonincreasing. Since ϕ is convex, it is easy to check that a mapping $T : D \to E$ is firmly nonexpansive if and only if

$$\|Tx - Ty\| \leq \|(1-t)(x-y) + t(Tx - Ty)\| \quad (3.19)$$

for each x and y in D and $t \in [0,1]$. It is clear that every firmly nonexpansive mapping is nonexpansive (cf. [5, 6]).

The following result extends a Lions-type iterative scheme [11] with condition (1.8) to a Banach space setting.

Corollary 3.6. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$ and let C be a nonempty closed convex subset of E. Let $T_1,\dots,T_N$ be firmly nonexpansive mappings from C into itself with $F := \cap_{i=1}^N \operatorname{Fix}(T_i) \neq \emptyset$ and*

$$F = \operatorname{Fix}\left(T_N \cdots T_1\right) = \operatorname{Fix}\left(T_1 T_N \cdots T_3 T_2\right) = \cdots = \operatorname{Fix}\left(T_{N-1} T_{N-2} \cdots T_1 T_N\right). \tag{3.20}$$

Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies conditions (1.2), (1.3), and (1.8). Then the iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto F.

Remark 3.7. (1) In Hilbert spaces, Lions [11, Théorèm 4] had used

(L1) $\lim_{n\to\infty} \lambda_n = 0$,

(L2) $\sum_{k=1}^{\infty} \lambda_{kN+i} = \infty$ for all $i = 0,\dots,N-1$, which is more restrictive than (1.3),

(L3)′ $\lim_{k\to\infty}\left(\sum_{i=1}^N |\lambda_{kN+i} - \lambda_{(k-1)N+i}| / (\sum_i^N \lambda_{kN+i})^2\right) = 0$ in place of (1.6).

(2) In general, (1.6) and (L3)′ are independent, even when $N = 1$. For more details, see [1].

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